Non-Perturbative Methods in 1+1D Field Theories

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Contents

1 Useful References

These notes we will cover, at a rather cursory level, a number of non-perturbative methods for tackling quantum systems where particles are restricted to move in one spatial dimension. This is quite a large, and active, subject of contemporary research. A number of excellent text books and articles cover many of these topics in considerable depth, to which I refer the reader. I have used a number of these heavily for inspiration for these notes (mistakes are, of course, my own).

- **Bosonization**

T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2003)
This is one of the standard references, and introduces bosonization first in a phenomenological manner, followed by an operator-led constructive approach.

A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 2004)
Another standard reference, this time introducing bosonization from a field theory led perspective. Also covers non-Abelian bosonization

D. Sénéchal, *An introduction to bosonization*, arXiv:cond-mat/9908262 (1999)
Taking a purely field theory approach, these notes are both detailed and clear.

J. von Delft and H. Schoeller, *Bosonization for Beginners — Refermionization for Experts*, Ann. Phys. (Berlin) **7** 225 (1998)
If ever you want to go through the operator-led constructive formalism in all its gory details, these are the notes for you.

- **Truncated Spectrum Methods**

V. P. Yurov and Al. B. Zamolodchikov, *Truncated conformal space approach to scaling Lee-Yang model*, Int. J. Mod. Phys. A **05**, 3221 (1990); *Truncated-fermionic-space approach to the critical 2D Ising model with magnetic field*, Int. J. Mod. Phys. A **06**, 4557 (1991)
This pair of articles first introduced the truncated conformal space approach and a free fermion formulation. Detailed and readable.

A. J. A. James, R. M. Konik, P. Lecheminant, N. J. Robinson, and A. M. Tsvelik, *Non-perturbative methodologies for low-dimensional strongly-correlated systems: From non-abelian bosonization to truncated spectrum methods*, Rep. Prog. Phys. **81**, 046002 (2018)
One of the only reviews of truncated spectrum methods, also covers non-Abelian bosonization and some other nonperturbative techniques.

- **Conformal Field Theory**

P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, 1996)
Often referred to as the "Big Yellow Book", this is the conformal field theory bible, covering almost everything one would need to know.

J. Cardy, *Conformal Field Theory and Statistical Mechanics*, arXiv:0807.3472 (2008)
Beautiful, clear lecture notes.

- **The Bethe Ansatz**

  V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993)
  The standard reference for integrability and the Bethe Ansatz. Perhaps not so accessible as a first reference.

  M. Gaudin, *The Bethe Wavefunction* (Cambridge University Press, 2014)
  Translated into English by J.-S. Caux, this is a beautiful introduction to the Bethe Ansatz

  F. Franchini, *An introduction to integrable techniques for one-dimensional quantum systems* (Springer, 2017); arXiv:1609.02100 (2016)
  Accessible introduction to integrability and the Bethe ansatz.

I will try to provide references to the appropriate literature along the way, although these will not be complete/thorough. Interested readers should explore the literature further!

2 Motivation

In a typical few course sequence on quantum field theory, it is rather easy to get the impression that diagrammatic perturbation theory is the be all and end all of the subject. Such a sequence usually starts by introducing a simple scalar field theory, such as $\lambda\phi^4$ in 3+1D, where one learns the ropes before building up to quantum electrodynamics and quantum chromodynamics. Along the way, a student becomes well-versed in the language and techniques of perturbation theory: renormalization and dimensional regularization to treat the divergences that appear, how to evaluate the various integrals that appear again and again via Feynman's tricks, and the philosophy of the renormalization group (in either its Wilsonian or Callan-Symanzik formulation). Often the only time non-perturbative methods are mentioned is in passing, perhaps in the form of instanton corrections[1,2] or a

[1] A. M. Polyakov, Phys. Lett. B **59**, 29 (1975)
[2] G.'t Hooft, Phys. Rev. D **14**, 3432 (1976).

brief mention of lattice gauge theory approaches.[3]

On the other hand, as a condensed matter physicist one is often studying quantum field theory in an attempt to describe the low-energy physics of large collections ($\sim 10^{23}$) of interacting electrons inside a given material. Details of the material, such as chemical composition and crystal structure, fix the form and strength of these interactions. From experiments, we know that these details really matter – we can realize a whole zoo of phases of matter with strikingly different properties. In many of the most fascinating materials (a well-known example would be the high-temperature cuprate superconductors[4]), the interactions are neither weak nor very strong, and there is *no small parameter in the problem*. We call these problems "strongly correlated" and their description is perhaps one of the grandest challenges in modern physics. What should an aspiring condensed matter field theorist do to tackle these problems?

The first option, for which there has been many successes, is to nonetheless adopt the tools of diagrammatic perturbation theory and treat interactions as being weak. This can sometimes be justified – despite the bare interaction being strong, the interaction term may be renormalization group irrelevant[5] and hence we are adiabatically connected to the noninteracting limit. (Most would not call such systems strongly correlated.) A famous example of such a scenario is the Landau-Fermi liquid.

A second option is to tackle the problem numerically. There are a diverse range of methods available, including: density functional theory (DFT), dynamical mean field theory (DMFT), exact diagonalization, the numerical renormalization group (NRG), quantum Monte Carlo (QMC), the density matrix renormalization group (DMRG), density matrix embedding theory, and tensor networks. Each of these has pros and cons. *Ab initio* methods, such as density function theory, allow one to treat details of realistic materials and complicated crystal structures, but they are approximate and struggle to describe interesting strongly correlated systems. Exact methods, such as exact diagonalization are limited to small systems, so finite size effects are large and momentum space resolution is low. Tensor network methods, such as the density matrix renormalization group (DMRG) in 1D and infinite projected entangled pair states (iPEPS) in 2D are powerful and can construct low-energy states to high accuracy in large (or infinite under some as-

[3] K. G. Wilson, Phys. Rev. D 10, 2445 (1974).

[4] See, e.g., E. Fradkin *et al.*, Rev. Mod. Phys. 87, 457 (2015).

[5] Here we use the condensed matter conventions: renormalization group relevant operators are those whose coupling flow to strong coupling as the ultraviolet cutoff is *decreased*. Conversely, irrelevant operators have couplings that flow to zero as the ultraviolet cutoff is decreased.

sumptions of structure of the state) systems, but struggle to construct eigenstates at finite energy density or strongly entangled states. Quantum Monte Carlo suffers with the well-known sign problem in fermionic or frustrated systems, but can access large system sizes and works well at high temperatures. It is still the case that numerical results in strongly correlated systems are debated and controversial, as well as challenging, even in very well studied systems and with the increased computational resources afforded by technological advances.

The third approach, which is pursued here, is to go in search of non-perturbative methods for tackling strongly correlated systems. In particular, we will turn our attention towards 1+1D quantum systems. It is not immediately clear that this is an easy regime to consider – if particles are confined to move on a line, exchange of particles necessitates scattering and thus interactions are inherently important and unavoidable. As a result, strong correlations and collective phenomena dominate the low-energy physics of 1+1D systems. But despite (or, indeed, because of) this, there are a number of exact results and methods peculiar to these systems that will help guide the way to tackling the physics of strong correlations. The hope is that insights gained in these special system may help us develop intuition that can be applied in a wider range of scenarios.

3 Why 1+1D?

In these notes we will focus on the case of quantum field theories with the fields living in one spatial and one temporal dimension (1+1D). The first point to address is then: Why are we restricting our attention to such theories?

- There exist *exact* solutions and *non-perturbative* methods for interacting quantum problems in 1+1D. This hopefully gives deep insights into the effect of interactions in higher dimensional quantum field theories.

- These theories are physically relevant in condensed matter physics. For example, there are materials where, for example through quirks of crystal structure, the degrees of freedom are confined to move and/or interact in only one spatial dimension. (A particular favorite of mine is the quasi-1D quantum magnet $CoNb_2O_6$, which has profound links to the exotic E_8 coset conformal field theory.[6]) There are also artificial quantum systems, such as

[6] R. Coldea *et al.*, Science **327**, 177 (2010).

ultracold atomic gases[7] or Josephson junction arrays[8], where the degrees of freedom are restricted to solely one spatial dimension.

- Interactions are inherently more important in lower dimensions. In the case of 1+1D systems, particles exchange *must* entail scattering. Strongly correlations often rule the roost in these problems.

- By studying 1+1D quantum field theories, we can gain insight into the properties of 2+0D classical problems (or vice versa, as perhaps best exemplified by the numerous works of Baxter[9] and works on the two-dimensional classical Ising model[10]).

- The treatment of these systems involves beautiful mathematics (quantum inverse scattering, conformal field theory, etc), which is quite satisfying.

3.1 Electrons in 1+1D. Let us being by considering a simple 1D chain of spin-1/2 fermions (e.g., electrons) described by the tight-binding Hamiltonian

$$H = -t\sum_{\ell=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left(c^\dagger_{\ell,\sigma}c_{\ell+1,\sigma} + \text{H.c.}\right) - \mu\sum_{\ell=1}^{L}\sum_{\sigma=\uparrow,\downarrow}c^\dagger_{\ell,\sigma}c_{\ell,\sigma}, \tag{1}$$

where t is the hopping amplitude, μ is the chemical potential (which sets the density of fermions), L is the number of lattice sites, and $c^\dagger_{\ell,\sigma}$ creates a fermion of spin $\sigma=\uparrow,\downarrow$ on the ℓth site of the lattice. We will consider periodic boundary conditions, $c_{L+1,\sigma} = c_{1,\sigma}$. The Hamiltonian can be diagonalized via Fourier transform

$$c_{\ell,\sigma} = \frac{1}{\sqrt{L}}\sum_{j=0}^{L-1} e^{ik_j\ell a_0}\tilde{c}_{k_j,\sigma}. \tag{2}$$

Here $k_j = 2\pi j/(La_0)$ is the quantized crystal momentum, a_0 is the lattice spacing, and $\tilde{c}_{k_j,\sigma}$ annihilates an electron with crystal momentum k_j and spin σ. In terms of momentum modes, the Hamiltonian reads

$$H = \sum_{j=0}^{L-1}\sum_{\sigma=\uparrow,\downarrow}\epsilon(k_j)\tilde{n}_\sigma(k_j), \tag{3}$$

[7] C. Gross and I. Bloch, Science **357**, 995 (2017).
[8] A. van Oudenaarden and J. E. Mooij, Phys. Rev. Lett. **76**, 4947 (1996).
[9] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Dover Publications, 1989).
[10] See, e.g., B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Mode* (Harvard University Press, 1973).

where we define the mode occupation numbers $\tilde{n}_\sigma(k_j) = \tilde{c}^\dagger_{k_j,\sigma}\tilde{c}_{k_j,\sigma}$ and the dispersion relation

$$\epsilon(k_j) = -2t\cos(k_j a_0) - \mu. \tag{4}$$

The ground state is thus defined by filling all modes k_j that satisfy $\epsilon(k_j) \leq 0$, which also defines the Fermi momentum

$$\pm k_F \quad : \quad \epsilon(\pm k_F) = 0. \tag{5}$$

For the model at hand, k_F can be expressed in terms of the density of electrons ρ_0 as

$$k_F = \frac{\pi\rho_0}{2}. \tag{6}$$

Let us now consider taking the continuum limit. If the Fermi momentum is away from the special points $k_F = 0, \pi/a_0$, we can proceed by linearizing the dispersion about the Fermi wave vector (and hence introducing a natural ultraviolet cutoff $\Lambda \sim k_F^2/2m$) by taking the leading order contribution to the Taylor expansion of $\epsilon(k_j)$ about $\pm k_F$. This is equivalent to splitting the spin-1/2 lattice fermion into left and right moving fields

$$c_{\ell,\sigma} \sim \sqrt{a_0}\Big[e^{ik_F x}R_\sigma(x) + e^{-ik_F x}L_\sigma(x)\Big], \tag{7}$$

where $x = \ell a_0$. Here we have factored out the fast oscillations, with wave vector $\pm k_F$, of the fermionic fields so that $R_\sigma(x), L_\sigma(x)$ are smooth, slowly-varying fields. The slowly varying fields satisfy the usual canonical anticommutation relations

$$\begin{aligned} &\Big\{R_\sigma(x), R^\dagger_{\sigma'}(y)\Big\} = \delta_{\sigma,\sigma'}\delta(x-y), \qquad \Big\{L_\sigma(x), L^\dagger_{\sigma'}(y)\Big\} = \delta_{\sigma,\sigma'}\delta(x,y), \\ &\Big\{L_\sigma(x), R^\dagger_{\sigma'}(y)\Big\} = 0. \end{aligned} \tag{8}$$

Here $\delta_{a,b}$ is the Kronecker delta function, whilst $\delta(x)$ is the Dirac delta function. In terms of the slowly varying left and right moving fields, the continuum Hamiltonian is

$$H = -iv_F \sum_\sigma \int \mathrm{d}x \left(R^\dagger_\sigma \partial_x R_\sigma - L^\dagger_\sigma \partial_x L_\sigma \right), \tag{9}$$

where $v_F = 2ta_0\sin(k_F a_0)$ is the Fermi velocity. Here we see that in order to keep the velocity finite (as it is in the lattice model), one must take the continuum limit as $a_0 \to 0$ and $t \to \infty$ with ta_0 fixed.

Equation (9) should hopefully be familiar as the Dirac Hamiltonian, containing fermions whose energy varies linearly with their momentum, $E(k) = \pm v_F k$. Thus the low-energy limit of the 1D lattice fermions, Eq. (1), is described by a relativistic theory! That is, we have an *emergent Lorentz invariance* (provided we ignore the existence of the ultraviolet cutoff). It turns out that this is a relatively common feature of electrons in 1+1D quantum systems,[11]. Notice that symmetry of the original lattice model (1), $U(1) \times SU(2)$, has been enhanced to $U(1)_R \times U(1)_L \times SU(2)$. Furthermore, this model exhibits a conformal symmetry (as will be discussed further in these notes).

The continuum Hamiltonian (9), and its simple generalization to the case with additional quantum numbers (such as electrons with spin and orbital indices),

$$H_{\rm gen} = -iv_F \sum_{\sigma=\uparrow,\downarrow} \sum_{\alpha=1}^{k} \int {\rm d}x \Big(R^\dagger_{\sigma,\alpha} \partial_x R_{\sigma,\alpha} - L^\dagger_{\sigma,\alpha} \partial_x L_{\sigma,\alpha} \Big). \tag{10}$$

will serve as the starting point for our subsequent discussions.

4 Abelian Bosonization

One of the keys to solving any problem in physics is identifying the correct degrees of freedom. These may be related in a simple manner to the original variables of the problem, or they may be some complicated combinations, through which a simpler description emerges. In condensed matter physics, our starting point is usually the electrons in a material, described via a fermionic field. In many problems of interest, these fields are strongly interacting and the excitations of the field become incoherent.[12] Extracting the physics of the problem then becomes very difficult, and instead we need to seek new variables in terms of which the physics is simple.

Abelian bosonization, the topic of this section, is one possible route for reformulating problems of interacting fermions. It identifies collective bosonic degrees of freedom, which are related in a non-local manner to the fermions, in terms of which the physics sometimes becomes particularly clear. For many problems of interacting fermions, the problems decouples into separate bosonic theories,

[11] See, e.g., A. M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics* (Cambridge University Press, 2010).

[12] This is another way of saying simple excitations of the electron quantum field have short life times.

each of which can be treated via integrability or semi-classical approximations.[13] As a result, in 1+1D Abelian bosonization has had many successes in describing the phenomenology of physically relevant strongly correlated electron and spin systems.

The fact that one can reformulate a theory of interacting fermions in terms of bosonic degrees of freedom (or vice versa) in one spatial dimension is not so surprising, if we think about it a little. Particles restricted to move in a single spatial dimension are rather special—exchange statistics and scattering phase shifts are inherently mixed—as exchanging the positions of two particles necessitates a scattering event. A simple case to consider is bosons that have a scattering phase shift of π – these will be described by a wave function that is *antisymmetric* under exchange of two bosons, precisely what would be expected from fermions.

From *Statistical Physics and Condensed Matter Theory I* you may already be familiar with a formulation of bosonic degrees of freedom (spin-1/2's) in terms of fermions: the Jordan-Wigner transformation.[14] This reads

$$\sigma^z_\ell = 2c^\dagger_\ell c_\ell - 1, \qquad \sigma^+_\ell = \exp\Big(i\pi\sum_{j<\ell} c^\dagger_j c_j\Big)c^\dagger_j, \qquad \sigma^-_\ell = \Big(\sigma^+_\ell\Big)^\dagger. \tag{11}$$

where σ^α are the Pauli spin operators and $c^\dagger_\ell$ creates a spinless fermion on site ℓ. These obey canonical anticommutation relations $\{c_j, c^\dagger_\ell\} = \delta_{j,\ell}$. The presence of the non-local exponential terms, known as a string operators,

$$\exp\Big(-i\pi\sum_{j<\ell} c^\dagger_j c_j\Big) \tag{12}$$

ensures the commutation relation of the spin operators on different sites $[\sigma^\alpha_j, \sigma^\beta_\ell] = 0$ if $j \neq \ell$‘.

Of course, it is easy to see from Eq. (11) that for certain problems such a reformulation doesn't help – one can still end up with a problem of strongly interacting fermions. Nevertheless, for other problems (such as the XY model[15]) can map to solvable problems of non-interacting fermions (this is a nice exercise to check yourself, see also Franchini's notes).

[13] See, e.g., the book by Gogolin, Nersesyan and Tsvelik, and the book by Giamarchi.

[14] Originally formulated in: P. Jordan and E. Wigner, Z. Phys. 47, 631651 (1928).

[15] The XY model Hamiltonian is: $H_{XY} = \sum_\ell J\Big[(1+\gamma)\sigma^x_\ell\sigma^x_{\ell+1} + (1-\gamma)\sigma^y_\ell\sigma^y_{\ell+1} + h\sigma^z_\ell\Big]$.

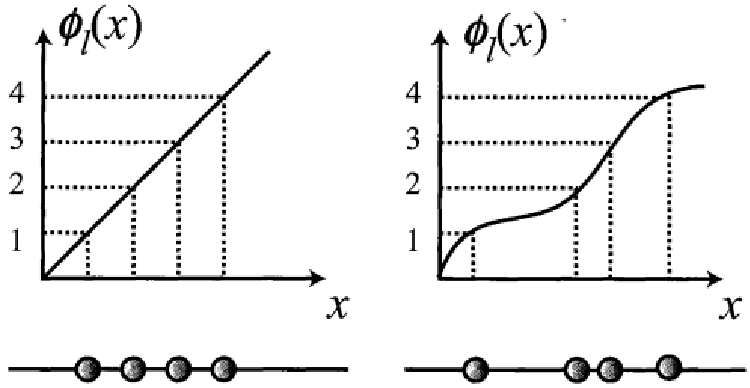

Figure 1: The mapping between the particle positions (lower panels) and the counting functions $\phi_l(x)$ (upper panels). Figure reproduced from T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University, 2003).

4.1 Phenomenological Abelian Bosonization. Before introducing the Abelian bosonization identities (a formal operator correspondence between fermionic and bosonic fields), we first phenomenologically motivate their form following the discussion of Giamarchi.

We begin by considering a one-dimensional system of particles (fermions or bosons) of average density ρ_0. Under first quantization the density is

$$\rho(x) = \sum_j \delta(x - x_j), \tag{13}$$

where the jth particle is found at position x_j (we order our particles such that $-\infty < x_1 < x_2 < x_3 < \ldots$ and so forth). The density can be reformulated in terms of a counting function $\phi_l(x)$, illustrated in Fig. 1, which takes values

$$\phi_l(x_j) = 2\pi j, \tag{14}$$

when its argument coincides with the position x_j of the jth particle. We choose $\phi_l(x)$ to herein be an always increasing function of x. Using properties of the Dirac delta function the density then becomes

$$\rho(x) = \sum_n \nabla\phi_l(x)\delta\Big(\phi_l(x) - 2\pi n\Big). \tag{15}$$

This can be further rewritten in Poisson sum form

$$\rho(x) = \frac{\nabla\phi_l(x)}{2\pi} \sum_p e^{ip\phi_l(x)}. \tag{16}$$

We see that the counting function $\phi_l(x)$ increases to $+\infty$ in the thermodynamic limit. This is not so pleasant, so we define a field $\phi(x)$ that measures the deviation from the perfect crystalline ordering of the particles

$$\phi_l(x) = 2\pi\rho_0 x - 2\phi(x). \tag{17}$$

Under this final reparameterization, the density becomes

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]\sum_p e^{2ip(\pi\rho_0 x - \phi(x))}. \tag{18}$$

Now, as the density at different places commutes it follows that $\phi(x)$ behaves similarly.

Let us now consider long wave length excitations[16] $q \sim 0$ – doing so is equivalent to averaging the density over distances large compared to ρ^{-1}. Doing this, Eq. (18) then becomes

$$\rho_{q\sim 0}(x) \sim \rho_0 - \frac{1}{\pi}\nabla\phi(x), \tag{19}$$

as we pick up only the $p = 0$ component.

We can now think about creation operators for the particles. These can, of course, be written as

$$\psi^\dagger(x) = [\rho(x)]^{\frac{1}{2}} e^{-i\theta(x)}, \tag{20}$$

where $\theta(x)$ is a bosonic field. To recover the (anti)commutation relations of the field $\psi(x)$, there must be non-trivial commutation relations between the density $\rho(x)$ and the field $\theta(x)$. For bosonic fields, denoted $\psi_B(x)$, we have

$$[\psi_B(x), \psi_B^\dagger(y)] = \delta(x-y), \tag{21}$$

which implies for $x = y$

$$\left[[\rho(x)]^{\frac{1}{2}}, e^{-i\theta(y)}\right] = 0, \quad x \neq y, \tag{22}$$

$$\left[\rho(x), e^{-i\theta(y)}\right] = \delta(x-y)e^{-i\theta(y)}. \tag{23}$$

Inserting Eq. (19), where one is using that $\phi(x), \theta(x)$ are slowly varying fields, one arrives at

$$\left[\frac{1}{\pi}\nabla\phi(x), \theta(y)\right] = -i\delta(x-y). \tag{24}$$

[16] This is reasonable if our continuum description is a low-energy effective description of a condensed matter system.

The higher harmonics $p > 0$ in Eq. (18) vanish (as shown in Giamarchi) when $x = y$, but odd harmonics remain for $x \neq y$. These higher harmonics are oscillatory with wave vector $2p\pi\rho_0$ and hence do not influence in the continuum limit (under the usual averaging arguments). Notice that Eq. (24) can be integrated by parts to tell us that the conjugate momentum $\Pi(x)$ to $\phi(x)$ is simply

$$\Pi(x) = \frac{1}{\pi}\nabla\theta(x). \tag{25}$$

Putting this all together, we obtain the creation operator for *bosonic particles* in Fig. 1 is

$$\psi_B^\dagger(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]^{1/2} \sum_p e^{2ip[\pi\rho_0 x - \phi(x)]} e^{-i\theta(x)}. \tag{26}$$

For fermionic particles, one needs to ensure that the fields anticommute. One can 'fix' the bosonic creation operator above by multiplying by the factor $\exp(i\phi_l(x)/2)$ which oscillates between ± 1 for consecutive particles (recall that $\phi_l(x_j)/2 = j\pi$). Then one has

$$\psi_F^\dagger(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]^{\frac{1}{2}} \sum_p e^{i(2p+1)[\pi\rho_0 x - \phi(x)]} e^{-i\theta(x)}. \tag{27}$$

Taking only the leading $p = 0, -1$ harmonics, we have

$$\psi_F^\dagger(x) \approx \left(\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right)^{\frac{1}{2}} \left(e^{-i\pi\rho_0 x} e^{i\phi(x)} e^{-i\theta(x)} + e^{i\pi\rho_0 x} e^{-i\phi(x)} e^{-i\theta(x)}\right). \tag{28}$$

This should be compared to Eq. (7), with Eq. (6) in mind, to arrive at phenomenological argument for the form of the bosonization identities

$$R(x) \propto e^{i\phi(x)} e^{-i\theta(x)}, \quad L(x) \propto e^{-i\phi(x)} e^{-i\theta(x)}. \tag{29}$$

In the following section we will discuss the rigorous operator correspondence, which indeed does look like the above.

4.2 The Abelian Bosonization Identities. In the previous section, we have seen that phenomenological arguments lead to a relatively simple operator correspondence between fermionic and bosonic fields. At heart, bosonization describes a formal correspondence between operators (or fields) in a fermionic theory with fields in a bosonic theory. The left- and right-mover fields ($L_\sigma(x)$ and $R_\sigma(x)$) of Eq. (7) are related to chiral bosonic fields via the so-called *bosonization identities*[17]

$$R_\sigma(x) \sim \frac{\eta_\sigma}{\sqrt{2\pi}} : e^{i\varphi_\sigma(x)} : \qquad L_\sigma(x) \sim \frac{\eta_\sigma}{\sqrt{2\pi}} : e^{-i\bar{\varphi}_\sigma(x)} : \tag{30}$$

The factors on the right hand side, exponentials of bosonic fields, are often called "vertex operators" from the string theory literature. We'll adopt this nomenclature herein. We will now discuss a number of technical points relevant to Eq. (30) (we follow the same lines as Sénéchal,[18] but with different normalization conventions for the fields/commutators).

4.2.1 Anticommutation Relations

Firstly, in a system of spin-1/2 fermions, there are two anticommuting species of fermions [cf. Eq. (8)]. To enforce the anticommutation of difference spin species, it is necessary to introduce "Klein factors", η_σ. These satisfy the anticommutation relations

$$\Big\{\eta_\sigma, \eta_{\sigma'}\Big\} = 2\delta_{\sigma,\sigma'}. \tag{31}$$

In order that left/right movers of the same spin species anticommute ($x \neq y$), we must have non-trivial commutation related for the chiral fields φ_σ and $\bar{\varphi}_\sigma$. This can be seen from the Campbell-Baker-Hausdorff formula

$$e^A e^B = e^B e^A e^{[A,B]}, \qquad \text{for } [A,B] = \text{const}. \tag{32}$$

and is similar to the discussion we had in the previous section for the Φ and Θ fields. Working through the Campbell-Baker-Hausdorff formula, we can obtain

$$\Big[\varphi_\sigma(x), \varphi_{\sigma'}(y)\Big] = -i\pi\delta_{\sigma,\sigma'}\mathrm{sgn}(x-y), \tag{33}$$

$$\Big[\bar{\varphi}_\sigma(x), \bar{\varphi}_{\sigma'}(y)\Big] = i\pi\delta_{\sigma,\sigma'}\mathrm{sgn}(x-y), \tag{34}$$

$$\Big[\varphi_\sigma(x), \bar{\varphi}_{\sigma'}(y)\Big] = -i\pi\delta_{\sigma,\sigma'}. \tag{35}$$

where minus signs are a matter of convention.

[17] It is worth noting that there are many different conventions for the bosonization identities, including the normalization of the bosonic fields, their commutation relations, where minus signs live, etc. A translation dictionary for some of the more common conventions can be found in the appendix of Giamarchi.

[18] D. Sénéchal, arXiv:cond-mat/9908262 (1999), where further details can be found.

Such non-trivial commutation relations between the fields comes from dealing properly with the "zero mode" of the bosonic field, which renders the usual mode expansion of the field and the conjugate momentum ill-defined. We briefly discuss this in the next section.

4.2.2 Mode Expansion for the Bosonic Field

We are used to writing a mode expansion for bosonic fields (and the conjugate momentum) that looks like

$$\Phi(x) \propto \int \frac{\mathrm{d}k}{2\pi} \sqrt{\frac{v}{2\omega(k)}} \Big[b(k) e^{ikx} + b^\dagger(k) e^{-ikx} \Big], \tag{36}$$

$$\Pi(x) \propto \int \frac{\mathrm{d}k}{2\pi} \sqrt{\omega(k)} 2v \Big[-ib(k) e^{ikx} + ib^\dagger(k) e^{-ikx} \Big]. \tag{37}$$

One can then separate into left ($k < 0$) and right ($k > 0$) moving degrees of freedom.

For bosonization, there are a couple of points here to note. Firstly, for the massless boson this mode expansion of the fields is ill-defined as $\omega(k=0) = 0$. Thus one needs to treat carefully, and separately, the zero mode. It is then not obvious that with such a zero mode one should get good left/right separation (actually, this is the origin of the nontrivial commutation relations in the previous section). For rigorous definitions, the field Φ must be compact – $\Phi \sim \Phi + 2\pi R$ are identified.

Putting these together, one gets the 'improved' (correct) mode expansion

$$\Phi(x,t) = q + \frac{\pi_0 vt}{L} + \frac{\bar{\pi}_0 x}{L} + \sum_{n>0} \frac{1}{\sqrt{4\pi n}} \Big[b_n e^{-kz} + b_n^\dagger e^{kz} + \bar{b}_n e^{-k\bar{z}} + \bar{b}_n^\dagger e^{k\bar{z}} \Big], \tag{38}$$

where

$$z = -i(x - vt) \equiv v\tau - ix, \qquad \bar{z} = i(x + vt) \equiv v\tau + ix, \tag{39}$$

where $\tau = it$ is imaginary time. Note that

$$\partial_z = -\frac{i}{2}\left(\frac{1}{v}\partial_t - \partial_x\right), \quad \partial_x = -i(\partial_z - \partial_{\bar{z}}), \tag{40}$$

$$\partial_{\bar{z}} = -\frac{i}{2}\left(\frac{1}{v}\partial_t + \partial_x\right), \quad \partial_i = iv(\partial_z + \partial_{\bar{z}}). \tag{41}$$

The canonical variables q and π_0 are the zero-mode, which has been factored out of the mode expansion, and the momentum of the mode operators is $k = 2\pi n/L$

where n is the integer summed. Operators obey the commutation relations

$$[b_n, b_m^\dagger] = 4\pi\delta_{n,m}, \quad [\bar{b}_n, \bar{b}_m^\dagger] = 4\pi\delta_{n,m}, \quad [q, \pi_0] = 4i\pi. \tag{42}$$

Separation into left- and right-moving fields can still be performed

$$\varphi(x,t) = Q + \frac{P}{2L}(vt - x) + \sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left(b_n e^{-kz} + b_n^\dagger e^{kz} \right), \tag{43}$$

$$\bar{\varphi}(x,t) = \bar{Q} + \frac{\bar{P}}{2L}(vt + x) + \sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left(\bar{b}_n e^{-k\bar{z}} + \bar{b}_n^\dagger e^{k\bar{z}} \right). \tag{44}$$

Here we define left/right zero modes

$$Q = \frac{1}{2}(q - \bar{q}), \qquad P = \pi_0 - \bar{\pi}_0, \qquad [Q, P] = 4\pi i, \tag{45}$$

$$\bar{Q} = \frac{1}{2}(q + \bar{q}), \qquad \bar{P} = \pi_0 + \bar{\pi}_0, \qquad [\bar{Q}, \bar{P}] = 4\pi i. \tag{46}$$

4.2.3 Normal Ordering

With the mode expansions introduced, we can discuss the normal ordering introduced in Eq. (30) via the notation $: O :$. Normal ordering allows us to deal with the pathologies that arise when the ultraviolet cutoff is taken to infinity, $\Lambda \to \infty$.[19] Normal ordering is one way to consistently deal with such problems. It involves expressing any bosonic field Φ as a mode expansion and then ordering the expression such that all boson annihilation operators are to the right of creation operators. For the fields discussed above, this amounts to

$$: e^{i\beta\varphi(x-vt)} := e^{i\beta Q} \exp\left[i\beta \sum_{n>0} \frac{1}{\sqrt{4\pi n}} b_n^\dagger e^{kz} \right] \exp\left[i\beta \sum_{n>0} \frac{1}{\sqrt{4\pi n}} b_n e^{-kz} \right] e^{-i\beta P(x-vt)/2L}. \tag{47}$$

In reality, we rarely need to use this identity. Instead, it is more important to understand how the vertex operators fuse together when normal ordered (i.e., how do we combine normal ordered exponentials?). Working through tedious algebra, left as an exercise, one finds

$$: e^{i\alpha\varphi(z)} :: e^{i\beta\varphi(z')} :=: e^{i\alpha\varphi(z) + i\beta\varphi(z')} : e^{-\alpha\beta\langle\varphi(z)\varphi(z')\rangle}. \tag{48}$$

[19] It is not so surprising that such pathologies can emerge. We have taken a system of lattice fermions, of finite band width, and approximated them by fermions with a linear dispersion (implicitly with a cutoff). If we take this cutoff to ∞, we then have to deal with having an infinite number of electrons below the Fermi level. This can introduce unphysical divergences, and one has to take care to remove them.

For the normalization of the bosonic fields that we are using, the Green's function for the bosons is

$$\langle \varphi(z)\varphi(0)\rangle = -\ln(z), \qquad \langle \bar{\varphi}(\bar{z})\bar{\varphi}(0)\rangle = -\ln(\bar{z}) \tag{49}$$

in the infinite volume $L \to \infty$ limit. Thus our normal-ordered vertex operators fuse as

$$: e^{i\alpha\varphi(z)} :: e^{i\beta\varphi(z')} :=: e^{i\alpha\varphi(z)+i\beta\varphi(z')} : (z-z')^{\alpha\beta}. \tag{50}$$

For normal ordered vertex operators, the expectation value on the vacuum state

$$\langle : e^{i\alpha\varphi(z)+i\beta\varphi(z')} : \rangle \tag{51}$$

vanishes unless the *neutrality condition* is met:

$$\langle 0| : e^{i\alpha\varphi(z)+i\beta\varphi(z')} : |0\rangle = \begin{cases} 0 & \text{if } \alpha+\beta \neq 0 \\ 1 & \text{otherwise.} \end{cases} \tag{52}$$

This follows from the properties of the zero mode:

$$\langle 0| e^{i(\alpha+\beta)Q} e^{-i(\alpha+\beta)P/2L} |0\rangle, \tag{53}$$

as P acts as an annihilation operator on the vacuum state (with its momentum is zero). Q is then a wildly fluctuating field (cf. the commutation relations of P and Q), and hence the expectation value of the vertex operator Q averages to zero. Note that this neutrality condition implies that for an operator O expressed in terms of vertex functions, the two-point function $\langle O^\dagger(z)O(z')\rangle$ can be non-zero.

Under our definition, as is implicit in Eq. (30), the vertex operators carry non-zero scaling dimension (this is often called the conformal field theory normalization). In particular, from the fusion rules and the bosonization identities, we have

$$\Big[\exp\big(i\alpha\varphi(z)\big)\Big] : \quad \Delta = \tfrac{\alpha}{2}, \quad \bar{\Delta} = 0, \tag{54}$$

$$\Big[\exp(i\bar{\alpha}\bar{\varphi}(\bar{z}))\Big] : \quad \bar{\Delta} = \tfrac{\bar{\alpha}}{2}, \quad \Delta = 0. \tag{55}$$

It is also worth recalling that the Green's function of non-interacting fermions is

$$\langle R^\dagger(x,t)R(0,0)\rangle = \frac{1}{2\pi z}, \qquad \langle L^\dagger(x,t)L(0,0)\rangle = \frac{1}{2\pi\bar{z}}. \tag{56}$$

Thus Eqs. (30) when combined with Eq. (50) correctly reproduce the fermion Green's functions.

4.2.4 Non-chiral Bosonic Fields

For relating back to the section on phenomenological bosonization, it is worth noting that the chiral fields φ_σ and $\bar{\varphi}_\sigma$ can be easily related to non-chiral ones

$$\varphi_\sigma(x) = \phi_\sigma(x) - \theta_\sigma(x), \qquad \bar{\varphi}_\sigma(x) = \phi_\sigma(x) + \theta_\sigma(x). \tag{57}$$

With this replacement, the bosonization identities (30) become

$$R_\sigma(x) \sim \frac{\eta_\sigma}{\sqrt{2\pi}} : e^{i(\phi_\sigma(x)-\theta_\sigma(x))} :, \qquad L_\sigma(x) \sim \frac{\eta_\sigma}{\sqrt{2\pi}} : e^{-i(\phi_\sigma(x)+\theta_\sigma(x))} : . \tag{58}$$

Comparing these to the formulae proposed at the end of the phenomenological bosonization section, see Eq. (29), we see that indeed they are of the same form. Sometimes it can be more convenient to work with non-chiral fields: for example, in the case with open boundary conditions.[20]

4.2.5 Formal Proof of the Correspondence

Formal proof of the correspondence between bosonic and fermionic free theories can be found in the Big Yellow Book and the notes of Sénéchal. They show that not only is there an operator correspondence, but the correspondence holds at the level of coincidence of the partition functions of the fermionic and bosonic theory. Thus the theory of free fermions and the free boson is in one-to-one correspondence.

4.3 Bosonizing Free Electrons. Having discussed some aspects of the formal correspondence, and the bosonization identities (30), let us move on to seeing how this works in practice. We will first consider the simple case of bosonizing a model of non-interacting electrons, before considering the case with interactions in the next section. In this section we'll see how the correspondence and transformations work in practice.

4.3.1 The Fermion Density Operator

Let us start by considering the density operators. For right movers, this reads

$$J_\sigma(x,t) = R^\dagger_\sigma(x,t) R_\sigma(x,t). \tag{59}$$

[20] This is natural, as one breaks the separation of left and right moving fields. This can be seen in a hand-waving manner by considering a left moving excitation incident on the left boundary – it scatters and turns into a right moving excitation. Thus there must be a term in the Hamiltonian, due to the boundary, that couples left- and right-moving chiral bosonic fields.

We can then insert Eq. (30), deal with the normal ordering, and proceed. Alternatively we can use *point splitting*, a more elegant and shorter way, to obtain the relevant operator result. We instead define

$$J_\sigma(z) = \lim_{\epsilon\to 0}\Big[R^\dagger_\sigma(z+\epsilon)R_\sigma(z) - \langle R^\dagger_\sigma(z+\epsilon)R_\sigma(z)\rangle\Big], \tag{60}$$

with ϵ have time-component > 0 to avoid issues with time-ordering. Applying ((30)) and ((49)) we have

$$J_\sigma(z) = \lim_{\epsilon\to 0}\Big[: e^{i\varphi_\sigma(z+\epsilon)} :: e^{-i\varphi_\sigma(z)} : -\frac{1}{\epsilon}\Big] \tag{61}$$

Fusing the vertex operators using Eq. (50) we then have

$$J_\sigma(z) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\Big[: e^{i\varphi_\sigma(z+\epsilon)-i\varphi_\sigma(z)} : \frac{1}{\epsilon} - \frac{1}{\epsilon}\Big]. \tag{62}$$

We can then perform a Taylor expansion of $\varphi(z+\epsilon)$ about z to get

$$\varphi_\sigma(z+\epsilon) - \varphi_\sigma(z) = \epsilon\partial_z\varphi_\sigma(z), \tag{63}$$

and expand the exponential as a power series in ϵ to get

$$J_\sigma(z) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\Big[\frac{1}{\epsilon}\Big(1 + i\epsilon\partial_z\varphi_\sigma(z) + O(\epsilon^2)\Big) - \frac{1}{\epsilon}\Big]. \tag{64}$$

We see the divergence in $1/\epsilon$ is removed in the point splitting procedure, yielding

$$J_\sigma(z) \equiv R^\dagger_\sigma(z)R_\sigma(z) = \frac{i}{2\pi}\partial_z\varphi_\sigma(z), \qquad \bar{J}_\sigma(\bar{z}) \equiv L^\dagger_\sigma(\bar{z})L_\sigma(\bar{z}) = -\frac{i}{2\pi}\partial_z\bar{\varphi}_\sigma(z). \tag{65}$$

Here the second equation follows from a similar calculation for left-moving fields.

4.3.2 The Fermion Kinetic Term

Having established how the density operator maps to bosonic operators, let us now consider the kinetic term that appears in the Hamiltonian (9). We will also evaluate this via point splitting

$$R^\dagger_\sigma\partial_x R_\sigma = -i\lim_{\epsilon\to 0}\Big[R^\dagger_\sigma(z+\epsilon)\partial_z R_\sigma(z) - \langle R^\dagger_\sigma(z+\epsilon)\partial_z R_\sigma(z)\rangle\Big] \tag{66}$$

Using the following expansion (to order ϵ^2s) for the fermion bilinear

$$R^\dagger_\sigma(z')R_\sigma(z) \; = \; \frac{1}{2\pi\epsilon} + \frac{i}{2\pi}\partial_z\varphi_\sigma(z) - \frac{\epsilon}{4\pi}\Big(\partial_z\varphi_\sigma(z)\Big)^2 - \frac{i\epsilon}{4\pi}\partial_z^2\varphi_\sigma(z) + O(\epsilon^2) \tag{67}$$

with $\epsilon = z' - z$, and differentiating with respect to z, we find

$$R_\sigma^\dagger \partial_z R_\sigma = \frac{1}{2\pi\epsilon^2} + \frac{1}{4\pi}\Big(\partial_z\varphi_\sigma\Big)^2 - \frac{3i}{2\pi}\partial_z^2\varphi_\sigma + \frac{\epsilon}{4\pi}\Big(i\partial_z^3\varphi_\sigma - 2\partial_z\varphi_\sigma\partial_z^2\varphi_\sigma\Big). \tag{68}$$

The point-splitting procedure will remove the divergence piece, and we are left with a single term which is not a total derivative (i.e. which integrates up to an assumed vanishing boundary term):

$$-i\int \mathrm{d}x\, R_\sigma^\dagger \partial_x R_\sigma \sim \frac{v}{4\pi}\int \mathrm{d}x \Big(\partial_z\varphi_\sigma\Big)^2 \tag{69}$$

Together with the left-moving piece, Eq. (9) becomes

$$H = \frac{v}{4\pi}\int \mathrm{d}x \sum_\sigma \left[\Big(\partial_z\varphi_\sigma\Big)^2 + \Big(\partial_{\bar z}\bar\varphi_\sigma\Big)^2\right]. \tag{70}$$

This is the free boson Hamiltonian. Hence we see that non-interacting fermions with linear dispersion become free bosons following the transformation (30).

4.4 The Hubbard Model: Bosonization. The one-dimensional Hubbard model, the simplest model of interacting electrons on a 1D lattice, is a nice example of a scenario where Abelian bosonization can be applied to predict and extract interesting and non-trivial physics. It is also a great test bed, as the one-dimensional model is integrable and can be solved via the nested Bethe ansatz.[21] The Hamiltonian reads

$$H_U = -t\sum_{\ell,\sigma}\Big(c^\dagger_{\ell,\sigma}c_{\ell+1,\sigma} + \mathrm{H.c.}\Big) + U\sum_\ell n_{\ell,\uparrow}n_{\ell,\downarrow}, \tag{71}$$

where U is the so-called Hubbard interaction strength, t is the hopping amplitude, and $n_{\ell,\sigma} = c^\dagger_{\ell,\sigma}c_{\ell,\sigma}$ is the number operator for spin-σ electrons on site ℓ of the lattice.

In the scaling limit, the interaction term can be split into three pieces, reflect-

[21] See F. H. L. Essler *et al.*, *The One-Dimensional Hubbard Model* (Cambridge University Press, 2010) for a comprehensive review of the exact solution, and its application to compute properties of the 1D Hubbard model.

ing different physics.

$$(i) \quad \sum_\sigma \left(R_\sigma^\dagger R_\sigma + L_\sigma^\dagger L_\sigma\right)\Big(R_{-\sigma}^\dagger R_{-\sigma} + L_{-\sigma}^\dagger L_{-\sigma}\Big)\,, \tag{72}$$

$$(ii) \quad \sum_\sigma \Big(R_\sigma^\dagger L_\sigma L_{-\sigma}^\dagger R_{-\sigma} + L_\sigma^\dagger R_\sigma R_{-\sigma}^\dagger L_{-\sigma}\Big)\,, \tag{73}$$

$$(iii) \quad \sum_\sigma \Big(e^{-i4k_F x} R_\sigma^\dagger L_\sigma R_{-\sigma}^\dagger L_{-\sigma} + e^{i4k_F x} L_\sigma^\dagger R_\sigma L_{-\sigma}^\dagger R_{-\sigma}\Big) \tag{74}$$

The first of these is the interaction term between "zero momentum" (non-oscillating) components of the lattice density operator. The second piece described interactions between the $2k_F$ and $-2k_F$ pieces of the density operator (such that the terms has net zero center of mass momentum). The final term describes scattering of two left-movers to two right-movers[22] and involves a change of momentum of $\pm 4k_F$. Generally such terms as suppressed, as they rapidly oscillate, except at special fillings (for example, at one electron per site $4k_F x = 2\pi j$ with $j \in \mathbb{Z}$ labeling lattice sites). We'll return to these umklapp terms later.

Applying the bosonization identities (30) to the Hubbard Hamiltonian (71) we find

$$H_U = \frac{v}{4\pi}\int \mathrm{d}x \sum_\sigma \left[\Big(\partial_x\varphi_\sigma\Big)^2 + \Big(\partial_x\bar\varphi_\sigma\Big)^2\right] + \frac{g}{(2\pi)^2}\int \mathrm{d}x \sum_\sigma \partial_x\varphi_\sigma\partial_x\bar\varphi_{-\sigma} \tag{75}$$

$$+\frac{g}{(2\pi)^2}\int \mathrm{d}x\bigg(\cos\left(\varphi_\uparrow - \varphi_\downarrow + \bar\varphi_\uparrow - \bar\varphi_\downarrow\right) - \cos\left(4k_F x + \varphi_\uparrow + \varphi_\downarrow + \bar\varphi_\uparrow + \bar\varphi_\downarrow\right)\bigg). \tag{76}$$

Here $g = Ua_0^2$, while the $4k_F x$ factor in the final cosine is understood as equaling $2\pi\mathbb{Z}$ when $4k_F = 2\pi/a_0$ and as leading the term to vanish otherwise (due to oscillatory terms being suppressed by the integration over x).

So far, the bosonized theory (76) looks like a mess, and it seems we've complicated the situation significant. However, all becomes clear when we change basis in terms of the bosons. This is suggested by the terms appearing in the cosine. We define chiral bosons

$$\varphi_c = \frac{1}{2}\left(\varphi_\uparrow + \varphi_\downarrow\right), \quad \varphi_s = \frac{1}{2}\left(\varphi_\uparrow - \varphi_\downarrow\right), \tag{77}$$

and similar for $\bar\varphi_c, \bar\varphi_s$ with $\varphi_\sigma \to \bar\varphi_\sigma$. The physical meaning of these fields is transport: φ_c describes 'charge' (density) degrees of freedom and φ_s describes

[22] This is known as umklapp scattering.

'spin' degrees of freedom. Further expressing theory in terms of non-chiral fields

$$\Phi_c = \varphi_c + \bar{\varphi}_c, \quad \Theta_c = \varphi_c - \bar{\varphi}_c, \tag{78}$$

and similar for Φ_s, Θ_s, we arrive at the low-energy effective description of the Hubbard model

$$H_U = \int \mathrm{d}x \Big(\mathcal{H}_c + \mathcal{H}_s \Big), \tag{79}$$

with

$$\mathcal{H}_c = \frac{v_c}{16\pi} \Big[K_c^{-1} (\partial_x \Phi_c)^2 + K_c (\partial_x \Theta_c)^2 \Big] - \frac{g}{(2\pi)^2} \cos(4k_F x + \Phi_c), \tag{80}$$

describing the charge bosonic fields, while the spin part is

$$\mathcal{H}_s = \frac{v_s}{16\pi} \Big[K_s^{-1} (\partial_x \Phi_s)^2 + K_s (\partial \Theta_s)^2 \Big] + \frac{g}{(2\pi)^2} \cos(\Phi_s). \tag{81}$$

In the charge sector, we have introduced the charge velocity v_c and the Luttinger parameter K_c,[23] which satisfy

$$v_c K_c^{-1} = v \left(1 + \frac{g}{(8\pi)^2} \right), \quad v_c K_c = \left(1 - \frac{g}{(8\pi)^2} \right). \tag{82}$$

The Luttinger parameter characterizes the interactions: for $K_c > 1$ they are effectively attractive and $K_c < 1$ they are repulsive. In the spin sector, we have introduced similar, but the situation is a little more complicated. Abelian bosonization obscures the non-Abelian SU(2) spin symmetry, which enforces $K_s = 1$.[24] We will come back to such problems in the section on non-Abelian bosonization.

4.5 The Bosonized Description: Extracting Physics. It worth emphasizing that the low-energy description of Eq. (79) is quite remarkable. It describes *decoupled* spin and charge degrees of freedom, despite the original model being built from electrons in which spin and charge are bound together. Unsurprisingly, this is known as *spin-charge separation* and is a fairly generic feature in

[23] Note there is also an annoying clash in conventions between different people as to whether K_c^{-1} sits in front of Φ_c or Θ_c. I stuck with my favorite convention, where $K_c < 1$ corresponds to effective repulsive interactions.

[24] This can be computed exactly, as the 1D Hubbard model is in fact exactly solvable via the Bethe ansatz, see the book by Essler, Frahm, Gohmann, Klümper and Korepin for all the details.

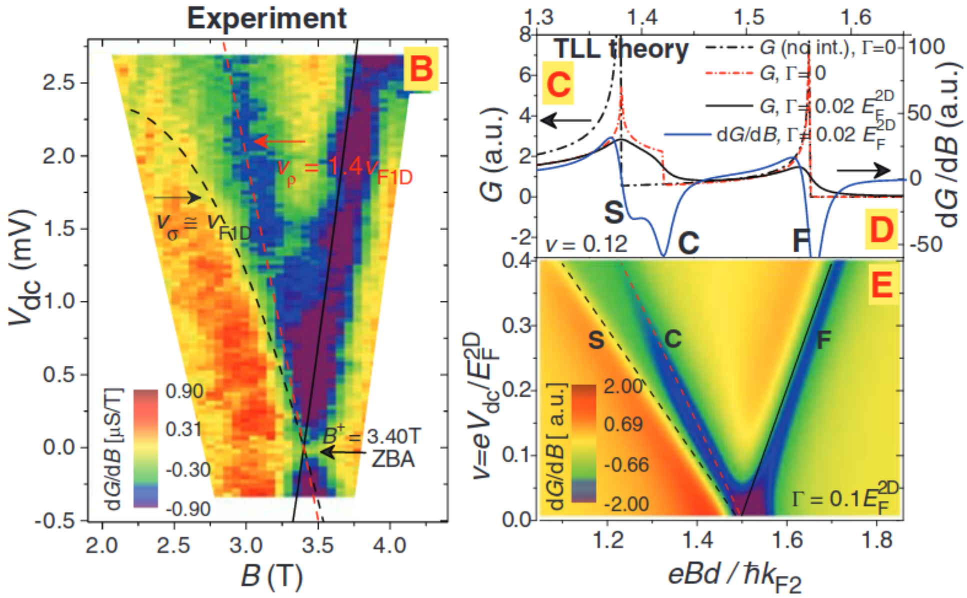

Figure 2: (Left) The differential conductance dG/dB as a function of magnetic field B and bias voltage V_{dc} measured in a one-dimensional quantum wire, created by electrostatically gating a 2D GaAs-AlGaAs quantum well. The black-dashed line arises from spinon excitations (which only carry spin quantum numbers), while the red-dashed line arises from holon excitations (which only carry charge). The solid black line is associated with an 'incipient' 2D dispersing excitation. On the right lower panel is shown the corresponding theory plot, with S and C showing spin and charge excitations, and F showing the 2D ones. Figure reproduced from Science **325**, 597 (2009).

1+1D theories. One can picture that the electron breaks apart (fractionalizes) into excitations that carry only spin or charge. These excitations propagate with different velocities, $v_c \neq v_s$ (generically), and this enables spin-charge separation to be observed in experiments on quasi-1D materials, see Fig. 2. In more complicated models where the electrons also carry orbital quantum numbers, one can find spin-charge-orbital separation, with excitations

Furthermore, there are a number of interesting points to immediately note. Firstly, let us consider changing the sign of the Hubbard interaction. Then $g \to -g$, and we see that when $4k_F x = 2\pi$ there is a *duality* between the charge and spin sectors. As we know that the spin sector of the lattice model carries an SU(2) symmetry, we have actually revealed the presence of unexpected SU(2) charge symmetry at half-filling ($4k_F = 2\pi/a_0$). Secondly, it is worth emphasizing that you might be familiar with the bosonic models we've found – each sector realizes the *sine-Gordon model*, a well-studied 1+1D integrable quantum field theory!

4.5.1 Generic Filling

Let us now briefly consider in more detail the physics that emerges for generic fillings, when $4k_F x \neq 2\pi\mathbb{Z}$. In this case the cosine term in the charge sector is suppressed by the $4k_F x$ oscillations, and it can be shown (see, e.g., Giamarchi's book) that the cosine in the spin sector is marginally irrelevant[25]. Then the low-energy physics of the Hubbard model is described by Gaussian bosonic theories for both the spin and charge sectors ($K_s = 1$ by SU(2) symmetry):

$$\mathcal{H} \Rightarrow \mathcal{H}_c^0 + \mathcal{H}_s^0, \tag{83}$$

$$\mathcal{H}_c^0 = \frac{\bar{v}_c}{16\pi}\Big[\bar{K}_c^{-1}(\partial_x\Phi_c)^2 + \bar{K}_c(\partial_x\Theta_c)^2\Big], \tag{84}$$

$$\mathcal{H}_s^0 = \frac{\bar{v}_s}{16\pi}\Big[(\partial_x\Phi_s)^2 + (\partial_x\Theta_s)^2\Big]. \tag{85}$$

Here $\bar{v}_{c,s}$ and $\bar{K}_c$ are the velocities and Luttinger parameter after running the renormalization group.

We have arrives at a theory with both gapless charge and spin degrees of freedom. With the charge sector being gapless, we have a metal – a small bias voltage induces current flow. This is in spite of the fact that we had strong electron-electron interactions: at generic fillings we nonetheless have metallic behavior. This helps explain the anomalously large conductivity observed in many 1+1D quantum systems.

With the low-energy description in terms of free bosons at hand, it is also easy to compute correlation functions of observables. Two-point functions are of particular interest, as those that decay slowest serve to characterize the phase.[26]

[25] I.e., when the interaction strength $g \ll 1$ and we can perturbatively integrate high energy modes (reducing the UV cutoff), we find that the scale-dependent interaction parameter $g(\Lambda)$ goes to zero as $\Lambda \to 0$.

[26] Note that 1+1D is again special: spontaneous breaking of a continuous symmetry is forbidden by the Mermin-Wagner theorem (see Phys. Rev. Lett. **17**, 1133 (1967)). Instead, quasi-long-range order is used to characterize phases. This is because the slowest decaying two-point functions lead to the most divergent susceptibilities, and thus if the continuous symmetry is broken external (for example, by an applied field) it will be this quasi-long-range order that develops in true long-range order.

Let us consider two examples of two-point functions, showing how bosonization quickly lets us examine the competition between phases, and allows us to extract the dominant fluctuations of a phase.

We consider two operators, associated with charge density waves (oscillations in the charge density) and superconducting pairing. In terms of lattice operators, we want to consider how the density and superconducting operators behave:

$$\begin{aligned}\sum_\sigma \sigma c^\dagger_{\sigma,j}c_{\sigma,j} \sim & a_0 \sum_\sigma \Big[R^\dagger_\sigma(x)R_\sigma(x) + L^\dagger_\sigma(x)L_\sigma(x)\Big] \\ & + a_0 \sum_\sigma \Big[e^{2ik_Fx}L^\dagger_\sigma(x)R_\sigma(x) + e^{-2ik_Fx}R^\dagger_\sigma(x)L_\sigma(x)\Big], \qquad (86)\\ c^\dagger_{\uparrow,j}c^\dagger_{\downarrow,j} \sim & a_0\Big[e^{-2ik_Fx}R^\dagger_\uparrow(x)R^\dagger_\downarrow(x) + e^{2ik_Fx}L^\dagger_\uparrow(x)L^\dagger_\downarrow(x)\Big] \\ & + a_0\Big[L^\dagger_\uparrow(x)R^\dagger_\downarrow(x) + R^\dagger_\uparrow(x)L^\dagger_\downarrow(x)\Big]. \qquad (87)\end{aligned}$$

In particular, as we interested in charge density waves we look at the oscillator ($2k_F$) piece of the density operator, and we focus on the zero momentum piece of the s-wave superconducting operator:

$$O_{2k_F}(x) = \sum_\sigma L^\dagger_\sigma(x)R_\sigma(x), \qquad O_{sSC}(x) = \sum_\sigma \sigma L^\dagger_\sigma(x)R^\dagger_{-\sigma}(x). \qquad (88)$$

Bosonizing both of these operators, and expressing them in terms of charge and spin degrees of freedom, we obtain

$$O_{2k_F}(x) = \frac{1}{2\pi}\sum_\sigma :e^{i(\Phi_c+\sigma\Phi_s)}:, \qquad O_{sSC}(x) = \frac{1}{2\pi}\sum_\sigma \sigma :e^{-i(\Theta_c+\sigma\Theta_s)}:. \qquad (89)$$

We can then compute two-point functions using that the theories are Gaussian. We do, however, have to deal with the Luttinger parameter in the charge sector, $\bar{K}_c$. If we define new fields

$$\Phi_c = \sqrt{\bar{K}_c}\bar{\Phi}_c, \qquad \Theta_c = \bar{\Theta}_c/\sqrt{\bar{K}_c}, \qquad (90)$$

then we recover the usual Gaussian model in terms of $\bar{\Phi}_c$ and $\bar{\Theta}_c$. Note that such a transformation also preserves the commutation relations of the bosonic fields. We can then compute everything as in a non-interacting Gaussian theory. The two-point functions of interest are:

$$\langle O^\dagger_{2k_F}(x)O_{2k_F}(0)\rangle \propto \langle :e^{i\sqrt{\bar{K}_c}\bar{\Phi}_c(x)}: e^{-i\sqrt{\bar{K}_c}\Phi_c(0)}:\rangle_c \langle :e^{i\sigma\Phi_s(x)}::e^{-i\sigma\Phi_s(0)}:\rangle_s, \qquad (91)$$

$$\langle O^\dagger_{sSC}(x)O_{sSC}(0)\rangle \propto \langle :e^{-i\bar{\Theta}_c(x)/\sqrt{\bar{K}_c}}: e^{i\Theta_c(0)/\sqrt{\bar{K}_c}}:\rangle_c \langle :e^{-i\sigma\Theta_s(x)}::e^{i\sigma\Theta_s(0)}:\rangle_s. \qquad (92)$$

Here we separated the two-point functions into separate spin and charge parts, using that the Hilbert space is a tensor sum of the two parts. Using the neutrality condition and the fusion rule Eq. (50) we arrive at

$$\langle O^\dagger_{2k_F}(x) O_{2k_F}(0)\rangle \propto \frac{1}{x^{1+\bar{K}_c}}, \qquad \langle O^\dagger_{sSC}(x) O_{sSC}(0)\rangle \propto \frac{1}{x^{1+1/\bar{K}_c}}. \tag{93}$$

As the slowest decaying correlation functions characterize the phase, we see that when the charge sector is effectively repulsive ($\bar{K}_c < 1$), the dominant quasi-long-range order is charge density wave, while when the interactions are effectively attractive ($\bar{K}_c > 1$) quasi-long range superconductivity wins.[27] Physically this seems very reasonable, and illustrates some of the phenomenological power of bosonization for capturing competition between different (quasi-long-range) orders.

4.5.2 Half Filling

What about when we do need to consider the $4k_F x = 2\pi\mathbb{Z}$ term? At half-filling, one electron per site, this term is present, and the Hamiltonian reads

$$\mathcal{H}_{hf} = \mathcal{H}_{c,hf} + \mathcal{H}_s, \tag{94}$$

$$\mathcal{H}_{c,hf} = \frac{v_c}{16\pi}\Big[K_c^{-1}(\partial_x\Phi_c)^2 + K_c(\partial_x\Theta_c)^2\Big] - \frac{g}{(2\pi)^2}\cos(\Phi_c), \tag{95}$$

$$\mathcal{H}_s = \frac{v_s}{16\pi}\Big[(\partial_x\Phi_s)^2 + (\partial_x\Theta_s)^2\Big] + \frac{g}{(2\pi)^2}\cos(\Phi_s). \tag{96}$$

The easiest way to extract the physics is via a renormalization group analysis. As already mentioned the cosine in the spin sector is marginally irrelevant, and the coupling flows to zero. In contrast, the sign difference in the charge sector leads to the cosine term being *marginally relevant* and it flows to strong coupling $\bar{g} \gg 1$. Thus in the low-energy limit, we have the effective Hamiltonian

$$\begin{aligned}\mathcal{H}_{\text{eff}} =& \frac{\bar{v}_c}{16\pi}\Big[\bar{K}_c^{-1}(\partial_x\Phi_c)^2 + \bar{K}_c(\partial_x\Theta_c)^2\Big] - \frac{\bar{g}}{(2\pi)^2}\cos(\Phi_c) \\ &+ \frac{v_s}{16\pi}\Big[(\partial_x\Phi_s)^2 + (\partial_x\Theta_s)^2\Big].\end{aligned} \tag{97}$$

[27] Notice that for superconductivity we require $\bar{K}_c > 1$, i.e. the low-energy effective interactions after the renormalization group flow are attractive. This does not necessarily imply that the initial Hubbard interaction U is attractive. Instead, it depends on the model-dependent renormalization group flow, under which the effective interactions can (in principle) change sign.

Further insight into the physics can be obtained by treating the cosine term semiclassically; the large prefactor $\bar{g}$ means that the cosine term 'pins' the bosonic field Φ_c to one of its minima

$$\Phi_c \to 2n\pi + \bar{\Phi}_c, \tag{98}$$

with small fluctuations about this minima being denoted $\bar{\Phi}_c$. To maintain the commutation relations, ordering of Φ_c results in Θ_c becoming completely incoherent. In terms of the low-energy effective theory we end up with an effective mass term for the fluctuations:

$$\mathcal{H}_{\rm eff} = \frac{\bar{v}_c}{16\pi}\Big[\bar{K}_c^{-1}(\partial_x\bar{\Phi}_c)^2 + \bar{K}_c(\partial_x\bar{\Theta}_c)^2\Big] + \bar{m}\bar{\Phi}_c^2 + \frac{v_s}{16\pi}\Big[(\partial_x\Phi_s)^2 + (\partial_x\Theta_s)^2\Big]. \tag{99}$$

Thus the charge sector is gapped (massive) and the system behaves as an insulator (moving charge around requires a finite energy $\bar{m}$). Thus at energies below the mass gap of the charge sector, $E \ll \bar{m}$, the low-energy physics is described solely in terms of spin degrees of freedom

$$\mathcal{H}_{\rm eff}\Big|_{E\ll\bar{m}} \approx \frac{v_s}{16\pi}\Big[(\partial_x\Phi_s)^2 + (\partial_x\Theta_s)^2\Big]. \tag{100}$$

This is an example of an interactions-driven metal-insulator transition, known in the literature as a Mott transition.[28]

In the Mott insulator phase, we have found a low-energy effective theory of gapless spin degrees of freedom. In other words, we obtain the low-energy theory of a gapless spin chain. Working through details (not shown here), this turns out to be the field theory for the scaling limit of the spin-1/2 Heisenberg model. This might ring a bell for you – it is precisely the result found on the lattice for the Hubbard model at half-filling and large U, where second order perturbation theory gives

$$H_U \to H_{\rm eff} = \frac{4t^2}{U}\sum_j \vec{S}_j\cdot\vec{S}_{j+1}. \tag{101}$$

As we already mentioned, it's not evident that $\mathcal{H}_s$ is the low-energy description of a spin chain with SU(2) symmetry. In the next section, we will discuss non-Abelian bosonization, where such identifications are clearer, but other aspects are trickier.

[28]Named after Sir Neville Mott, who first theoretically described this in Proc. Roy. Soc. A **62**, 416 (1949).

5 Non-Abelian Bosonization

5.1 Motivation. In the preceding section we saw that non-Abelian symmetries, such as SU(2) spin rotations, are hard to see in the Abelian bosonized description of a model. This is explicit from the bosonized expressions for the spin operators in the Mott insulator phase of the Hubbard model [cf. Eq. (101)], which read as

$$S^z_j = \frac{1}{2}c^\dagger_{j,\alpha}\sigma^z_{\alpha\beta}c_{j,\beta} = \frac{a_0}{2\sqrt{2}\pi}\partial_x\Phi_s(x) + \frac{\lambda(-1)^{x/a_0}}{2\pi}\sin\left(\frac{\Phi_s(x)}{\sqrt{2}}\right), \tag{102}$$

$$S^\pm_j = \frac{1}{2}c^\dagger_{j,\alpha}\left(\sigma^x \pm i\sigma^y\right)_{\alpha\beta}c_{j,\beta} = \frac{\lambda}{2\pi}e^{\mp i\Theta_s(x)/\sqrt{2}}\left[(-1)^{x/a_0} + \cos\left(\frac{\Phi_s(x)}{\sqrt{2}}\right)\right], \tag{103}$$

where λ is a non-universal parameter that is related to the expectation value $\langle\cos(\Phi_c/\sqrt{2})\rangle$ in the Mott insulating phase. It is more than apparent that this representation of the operators is not SU(2) symmetric, yet it must also be the case that if the theory *is* SU(2) symmetric then expectation values $\langle S^x_j S^x_l\rangle$ and $\langle S^z_j S^z_l\rangle$ are identical.[29] It is simply the case that the representation implemented hides this fact. A natural question to ask is then: Is there a reformulation of fermionic degrees of freedom in terms of bosonic ones that makes non-Abelian symmetries manifest? The answer is yes, via non-Abelian bosonization, and this will be the subject of this section of the notes.

5.2 The Kac-Moody Algebra. We take as our starting point the field theory of non-interacting fermions carrying both spin and orbital quantum numbers. We generalize Eq. (10) to consider the case with N spin indices

$$H = -iv_F\sum_{\alpha=1}^{k}\sum_{\sigma=1}^{N}\int \mathrm{d}x\,\left(R^\dagger_{\alpha,\sigma}\partial_x R_{\alpha,\sigma} - L^\dagger_{\alpha,\sigma}\partial_x L_{\alpha,\sigma}\right). \tag{104}$$

This Hamiltonian has $\mathbb{Z}_2\times\mathrm{U}(1)\times\mathrm{SU}(N)\times\mathrm{SU}(k)$ symmetry, corresponding to separate number conservation of left and right movers, spin rotations, and orbital rotations.

We can define SU(N) and SU(k) current operators via

$$J^a_R = R^\dagger\left(I\otimes s^a\right)R, \qquad a = 1,\ldots,N^2-1, \tag{105}$$

$$F^a_R = R^\dagger\left(t^a\otimes I\right)R, \qquad a = 1,\ldots,k^2-1, \tag{106}$$

[29] Showing that they are indeed the same is left as an exercise for the motivated reader.

where I is the identity matrix, s^a are generators of the su(N) algebra, t^a are generators of the su(k) algebra, and we use the short hand notation

$$R^\dagger(t^a \otimes s^b)R = \sum_{\alpha,\alpha'=1}^{k} \sum_{\sigma,\sigma'=1}^{N} R^\dagger_{\alpha\sigma} t^a_{\alpha\alpha'} s^b_{\sigma\sigma'} R_{\alpha'\sigma'}. \tag{107}$$

The generators of the su(N) algebra are normalized such that

$$\mathrm{tr}(s^a s^b) = \frac{1}{2}\delta_{a,b}, \qquad [s^a, s^b] = i f^{abc} s^c. \tag{108}$$

Here f^{abc} are the structure constants of the Lie algebra su(N). The su(k) generators are similarly normalized.

The commutation relations of the fermionic fields, together with those of the generators, gives us the commutation relations of the current operators. Since fields of different chiralities commute, only currents of the same chirality have non-trivial commutation relations. For the SU(N) currents these read:

$$\Big[J^a_\ell(x), J^b_\ell(y)\Big] = i f^{abc} J^c_\ell(x)\delta(x-y) - (-1)^\ell \frac{ik}{4\pi}\delta'(x-y)\delta_{a,b}. \tag{109}$$

Here $\ell = R, L = 0, 1$ denotes the chirality, $\delta'(x)$ is the derivative of the Dirac delta function, and k is the number of orbitals. This is the Kac-Moody algebra, and the currents $J^a_\ell(x)$ are said to be $\mathrm{SU}(N)_k$ currents (read as SU(N) level k currents). Similarly, the $F^a_\ell(x)$ are $\mathrm{SU(k)_N}$ currents.

5.2.1 Computing the Anomalous Commutator

The final term in Eq. (109) is called the anomalous commutator, or the Schwinger term. It can be obtained via the field theory definition of the commutator

$$\Big\langle \Big[J^a_\ell(x), J^b_\ell(y)\Big] \Big\rangle = \lim_{\tau\to 0^+} \Big\langle J^a_\ell(x,\tau)J^b_\ell(y,0) - J^a_\ell(x,-\tau)J^b_\ell(y,0) \Big\rangle, \tag{110}$$

and using the propagator

$$\Big\langle R_{\alpha\sigma}(x,\tau)R^\dagger_{\alpha'\sigma'}(x',\tau') \Big\rangle = \frac{1}{2\pi}\frac{\delta_{\alpha,\alpha'}\delta_{\sigma,\sigma'}}{(\tau-\tau') - i(x-x')}, \tag{111}$$

as follows:

$$\Big\langle \Big[J^a_R(x), J^b_R(y)\Big] \Big\rangle = \frac{k\delta_{a,b}}{2} \lim_{\tau\to 0^+} \frac{1}{4\pi^2}\left[\frac{1}{(\tau - i(x-y))^2} - \frac{1}{(\tau + i(x-y))^2}\right], \tag{112}$$

$$= \frac{k\delta_{a,b}}{8\pi^2}\partial_x\left(\frac{1}{x-y+i0^+} - \frac{1}{x-y-i0^+}\right), \tag{113}$$

$$= -\frac{ik}{4\pi}\delta'(x-y)\delta_{a,b}, \tag{114}$$

with appropriate sign changes for left-moving currents from the sign change in the propagator.

5.2.2 Fourier Components of the Currents

As usual with translationally invariant models, it can often be more convenient to work with Fourier modes (like the mode occupation numbers of a free electron gas). Placing our system into a box of volume l, Fourier components of the currents are defined through

$$J^a(x) = \frac{1}{l}\sum_{n=-\infty}^{\infty} e^{-2\pi i n x/l} J^a_n, \tag{115}$$

where J^a_n is the nth Fourier component and obeys the algebra

$$[J^a_n, J^b_m] = i f^{abc} J^c_{n+m} + \frac{nk}{2}\delta_{n+m,0}\delta_{a,b}. \tag{116}$$

Hopefully this looks somewhat familiar to you from the conformal field theory course. Notice that the zeroth Fourier component constitutes a subalgebra

$$[J^a_0, J^b_0] = i f^{abc} J^c_0, \tag{117}$$

which is isomorphic to the global algebra (108).

5.3 Conformal Embedding and Wess-Zumino-Novikov-Witten Models. Why have we gone to the trouble of introducing SU(N) and SU(k) currents obeying the Kac-Moody algebra (109)? Well, it turns out that a theory of free fermions with symmetries can be written in terms of a sum of Hamiltonians describing the different symmetry sectors. Each of these, in turn, can be written solely in terms of the associated currents. The Hamiltonians describing each symmetry sector are known as Wess-Zumino-Novikov-Witten (WZNW) models.[30] These are conformal, so the rules for fractionalizing the original Hamiltonian into separate Hamiltonians for each symmetry sector is often called a *conformal embedding*.

For the Hamiltonian in Eq. (10) the conformal embedding reads as

$$H = H[\mathrm{U}(1)] + W[\mathrm{SU}(N); k] + W[\mathrm{SU}(k); N], \tag{118}$$

[30]The correspondence between fermionic theories and Wess-Zumino-Witten models was first proved by E. Witten in Commun. Math. Phys. **92**, 455 (1984). This is a surprisingly readable paper on a very technical subject.

where $W[\mathrm{G};k]$ is the WZNW Hamiltonian for the group G at level k (cf. Eq. (109)). As already mentioned, it can be written in a particularly simple manner in terms of the G level k currents, known as the Sugawara form.[31] For SU(N) models these read

$$W[\mathrm{SU}(N);k] = \frac{2\pi}{N+k}\int_0^l \mathrm{d}x\,(:J^a_R J^a_R: + :J^a_L J^a_L:)\,, \tag{119}$$

$$= \frac{2\pi}{l(N+k)}\left(J^a_{R,0}J^a_{R,0} + 2\sum_{n>0} J^a_{R,-n}J^a_{R,n} + R \leftrightarrow L\right). \tag{120}$$

For the U(1) part, we have instead

$$H[\mathrm{U}(1)] = \frac{\pi}{Nk}\int \mathrm{d}x\,\left(:j_R^2: + :j_L:^2\right), \tag{121}$$

with the U(1) currents being

$$j_R =: R^\dagger_{\alpha\sigma}R_{\alpha\sigma}: \qquad j_L =: L^\dagger_{\alpha\sigma}L_{\alpha\sigma}:. \tag{122}$$

The conformal embedding, Eq. (118), tells us how the theory breaks up into degrees of freedom that solely carry quantum numbers associated to each of the symmetries of the theory. A (loose) analogy to this, which might be familiar to you, is how in classical mechanics we can decompose kinetic energy into radial and angular pieces

$$\frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2mr^2}\vec{L}^2. \tag{123}$$

Here the first term would correspond to the Gaussian theory, and the second is written in terms of a vector (the angular momentum).

Each of the terms appearing on the right hand side of the conformal embedding (118) commute with one-another. Eigenstates of the total Hamiltonian are those simultaneous eigenstates of each piece, and accordingly we can treat each symmetry sector separately. This is the analogue to the spin-charge separation that we encountered in the previous section on Abelian bosonization. Much like with the classical mechanical analogue, such a separation of degrees of freedom can be very useful: if there was a radially-symmetric potential in the classical problem, the identified degrees of freedom on the right hand side of Eq. (123) are, of course, going to be much easier to work with than those on the left-hand side.

[31] Such theories, composed solely of currents, where first study by Hirotaka Sugawara in the 1960s, see Phys. Rev. **170**, 1659 (1968), hence the name.

5.3.1 Differences between Abelian and non-Abelian Bosonization: Conformal Blocks

While the conformal embedding tells us how to split up the electron into other degrees of freedom, each of which transform separately under one of the symmetries of the theory. This is like spin-charge separation in Abelian bosonization. However, one should be a little careful about drawing intuition from Abelian bosonization and applying it to non-Abelian bosonization. We will illustrate this explicitly here by considering bosonization at the level of operators. For Abelian bosonization we have the bosonization identities (cf. Eq. (30)) of the form

$$R(x) \sim \frac{1}{\sqrt{2\pi}} : e^{i\varphi(x)} : \qquad L(x) \sim \frac{1}{\sqrt{2\pi}} : e^{-i\bar{\varphi}(x)} : \tag{124}$$

This illustrates a more general feature of Abelian bosonization: operators are expressed in terms of sums or products of local operators that act in a chiral $(\varphi, \bar{\varphi})$ sector of a Gaussian model (the free boson). This separation into chiral sectors is very convenient, as we saw in the previous section, but this is *not* a universal property of a conformal field theory.

In general, multi-point correlation functions of conformal field theories are not products of holomorphic functions, as implied by the bosonization identities above, instead they are sums of products of holomorphic functions

$$\langle O(z_1, \bar{z}_1) \dots O(z_N, \bar{z}_N) \rangle = \sum_a c_a F_a(z_1, \dots, z_N) \bar{F}_a(\bar{z}_1, \dots, \bar{z}_N). \tag{125}$$

Here the holomorphic functions, F_a and $\bar{F}_a$, are called conformal blocks and c_a are coefficients that are fixed by single-valuedness of the correlation function.

A (perhaps) familiar illustrative example is the four-point function of the spin operator in the Ising conformal field theory. This operator $\sigma(z, \bar{z})$ has conformal dimensions (1/16, 1/16) and operator product expansion

$$\sigma(z, \bar{z})\sigma(w, \bar{w}) \sim \frac{1}{|z-w|^{1/4}} + \frac{1}{2}|z-w|^{3/4}\epsilon(w, \bar{w}). \tag{126}$$

Applying this, the four point function is

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\sigma(z_3, \bar{z}_3)\sigma(z_4, \bar{z}_4) \rangle = \frac{1}{|z_{12}|^{1/4}|z_{34}|^{1/4}} \left(1 + \frac{1}{4} \frac{|z_{12}||z_{34}|}{|z_{24}|^2} \right), \tag{127}$$

where $z_{ij} = z_i - z_j$. This expression is clearly the sum of product of holomorphic functions.

5.4 Solving the WZNW Model Returning to the Sugawara form, Eq. (120), let us now discuss how to diagonalize the Hamiltonian. This is a relatively straight forward task, despite the conformal embedding looking, at first glance, rather complicated.

To begin, we note that (120) is formed from two commuting pieces, corresponding to left- and right-moving excitations

$$W[G;k] = H_R + H_L, \tag{128}$$

$$H_d = \frac{2\pi}{l(k+c_v)}\left[J^a_{d,0}J^a_{d,0} + 2\sum_{n>0} J^a_{d,-n}J^a_{d,n}\right]. \tag{129}$$

Here we write the more general form of the Hamiltonian in terms of the quadratic Casimir c_v in the adjoint representation ($f_{abc}f_{\bar{a}bc} = c_v\delta_{a,\bar{a}}$). The separation into commuting left/right moving pieces is reasonable, especially in light of the fact that we started from a theory a non-interacting massless fermions where right- and left-movers are independent. More generally, this decomposition of the Hilbert space is a general property of conformal field theories.[32] We can discuss the left/right piece separately due to this decomposition.

To continue, we construct the lowest eigenstates of (129) by starting from vacuum states $|h\rangle$ that are annihilated by the $n > 0$ components of the currents

$$J^a_n|h\rangle = 0, \quad n > 0. \tag{130}$$

These are then eigenstates of a "quantum spinning top" Hamiltonian

$$H_{\rm top} = \frac{2\pi}{l(k+c_v)}J^a_0J^a_0, \quad [J^a_0, J^b_0] = if^{abc}J^c_0. \tag{131}$$

Their energies are related to the quadratic Casimir invariants of the group $c_2[h]$;[33] for SU(N) we have

$$E[h] - E_0 = \frac{2\pi}{l}\frac{c_2[h]}{N+k}. \tag{132}$$

In the familiar case of SU(2), these states realize the irreducible representations of SU(2) and the quadratic Casimir invariant take values corresponding to the

[32] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).

[33] Consider a representation h of a group with generators $T^a[h]$. The quadratic Casimir operator is defined as $\hat{c}_2[h] = T^a[h]T^a[h]$. By construction this commutes with every element of the algebra and hence, via Schur's Lemma, $\hat{c}_2[h] = c_2[h]I$ with I the identity matrix. This defines the quadratic Casimir invariant $c_2[h]$.

total spin $c_2[h] = h(h+1)$, with $h = 1/2, 1, 3/2, \ldots$. States are characterized not only through h but by its projection $j^z = -j, -j+1, \ldots, j$ and hence the lowest energy states are degenerate.

Higher energy eigenstates are obtained from the states $|h, h^z\rangle$ by applying Kac-Moody currents with negative Fourier components

$$J^{a_1}_{-n_1} J^{a_2}_{-n_2} \ldots J^{a_p}_{-n_p} |h, h^z\rangle, \tag{133}$$

where n_j are positive integers. In the case of the $\mathrm{SU}(2)_k$ WZNW model these states have energies E given by

$$E - E_0 = \frac{2\pi}{l}\left[\frac{c_2[h]}{2+k} + \sum_{j=1}^{p} n_j\right]. \tag{134}$$

Thus we diagonalize the Hamiltonian, and have knowledge of both eigenstates and eigenvalues.

5.5 The Lagrangian Formulation of the WZNW model

5.6 An Example: Semi-classical Analysis of U(1)×SU(2)×SU(k) Model with $k \gg 1$

5.7 Other Applications. Non-Abelian bosonization has been applied to a wide range of problems in 1+1D. Some examples, beyond those discussed above, are:

1. *Spin chains and ladders.*— Non-Abelian bosonization has been extensively applied to describe the physics of spin chains and ladders, as summarized in a number of text books.[34] With its ability to make manifest non-Abelian symmetries, the non-Abelian bosonization framework is particular useful in these scenarios and helps make the physics more transparent. This approach was first developed and applied in a number of seminal works, including those of Polyakov and Wiegmann,[35] Affleck,[36] and Affleck and Haldane.[37] This still remains a fruitful area of research.

[34] See the textbook by Gogolin, Nersesyan and Tsvelik for one such example.
[35] A. M. Polyakov, P. B. Wiegmann, Phys. Lett. B **141**, 223 (1984).
[36] I. Affleck, Phys. Rev. Lett. **55**, 1355 (1985); Nucl. Phys. B **265**, 409 (1986).
[37] I. Affleck, F. D. M. Haldane, Phys. Rev. B **36**, 5291 (1987).

2. *The Kondo problem.*— The Kondo problem, a magnetic impurity interacting with a sea of electrons, is renowned in condensed matter physics for its non-perturbative phenomena. Famously Wilson developed the numerical renormalization group to tackle the problem, which later turned out to be exactly solvable due to integrability. Fradkin and collaborations[38] first showed that non-Abelian bosonization can be useful when applied to the Kondo problem. Subsequent works pursuing this approach include those of Affleck and Ludwig.[39] Generalizations of the Kondo problem to include higher numbers of conduction channels,[40] impurities with structure[41] and periodic arrangement of the impurities (known as the Kondo lattice)[42] can also be tackled within this framework.

3. *Disordered fermions.*— Another fruitful application of non-Abelian bosonization has been to theories with disorder, which posses a lot of interesting physics. It has been used to tackle a wide-range of problems, including: Dirac fermions in random non-Abelian gauge potentials,[43] disordered d-wave superconductors,[44] surface states of disordered topological superconductors,[45] non-Hermitian theories with random mass terms[46] or random potentials,[47] which have applications to percolation problems. This area of research also remains quite active.

4. *Quantum Hall transitions and edge states.*— There are also various applica-

[38] E. Fradkin, C. von Reichenbach, F. A. Schaposnik, Nucl. Phys. B **316**, 710 (1989).

[39] I. Affleck, Nucl. Phys. B **336**, 517 (1990); I. Affleck, A. W. W. Ludwig, *ibid.* **352**, 849 (1991); *ibid.* **360**, 641 (1991); Phys. Rev. Lett. **67**, 3160 (1991).

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[43] See, e.g., D. Bernard, arXiv e-prints (1995), hep-th/9509137; J.-S. Caux, I. I. Kogan, A. M. Tsvelik, Nucl. Phys. B **466**, 444 (1996); C. Mudry, C. Chamon, X.-G. Wen, *ibid.* **466**, 383 (1996).

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[45] See, for example, the works of Matt Foster and collaborators, including: M. S. Foster, E. A. Yuzbashyan, Phys. Rev. Lett. **109**, 246801 (2012); M. S. Foster, H.-Y. Xie, Y.-Z. Chou, Phys. Rev. B **89**, 155140 (2014); Y.-Z. Chou, M. S. Foster, *ibid.* **89**, 165136 (2014).

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[47] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, G. Grinstein, Phys. Rev. B **50**, 7526 (1994).

tions of non-Abelian bosonization to the quantum Hall effect, including transitions between quantum Hall states[47] and the description of quantum Hall edge state[48]. Coupled-wire constructions of two-dimensional non-Abelian fractional Hall states and chiral spin liquid phases also make copious use of the framework.[49]

5. *Quantum chromodynamics in 1+1D and the quark-gluon plasma in 3+1D.*— Non-Abelian bosonization has also been applied to problems in high-energy physics. Toy models of quantum chromodynamics in 1+1D, for example, have been studied within this approach,[50] as well as realistic models of dense quark-gluon plasma.[51]

6 Truncated Spectrum Methods

6.1 The Truncated Conformal Space Approach.

6.2 The Truncated Free Fermion Space Approach – Ising Field Theory.

6.3 Bootstrapping From Integrability.

7 A Few Concluding Words

In these notes...

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A A Conformal Field Theory Primer

Conformal field theory is a huge subject (see the "Big Yellow Book" for example[52]), with profound relations to pure mathematics and beautiful physics. It is not a subject that we can do justice to in these short notes, and is the topic of one of the MSc courses. Here we will simply introduce some of the useful (for these notes) concepts from conformal field theory in a merely "computational" manner.

B A Brief Overview of the Bethe Ansatz

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